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A Note on the Apostol Type q-Frobenius-Euler Polynomials and Generalizations of the Srivastava-Pinter Addition Theorems

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Abstract. The main subject of this study is to define and investigate for the Apostol type Frobenius-Euler polynomials. We give some identities for these polynomials. We generalize the Srivastava-Pintér addition theorems between the Bernoulli polynomials and Apostol type Frobenius-Euler polynomials.

1. Introduction, Definitions and Notations

Throughout this paper, we always make use of the following notation; $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The q-numbers and q-factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q}, q \neq 1,$$

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q ,$$

respectively, where $[0]_q! = 1$ and $n \in \mathbb{N}$, $a \in \mathbb{C}$. The q-binomial coefficient is defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{(q:q)_n}{(q:q)_{n-k}(q:q)_k}.$$

The q-analogue of the function $(x+y)_q^n$ is defined by

$$(x+y)_q^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{k(k-1)}{2}} x^{n-k} y^k.$$

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The q-binomial formula is known as

$$(1-a)_q^n = \prod_{j=0}^{n-1}\left(1-q^j a\right) = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}}(-1)^k a^k.$$

In the standard approach to the q-calculus two exponential functions are used

$$e_q(z) = \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty}\frac{1}{(1-(1-q)\,q^k z)},\ 0<\left|q\right|<1,\ |z|<\frac{1}{\left|1-q\right|}$$

and

$$E_q(z) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty}\left(1+(1-q)\,q^k z\right),\ 0<\left|q\right|<1,\ z\in\mathbb{C}.$$

From this form, we easily see that $e_q(z)E_q(-z)=1$. Moreover $D_q e_q(z) = e_q(z)$, $D_q E_q(z) = E_q(qz)$ where D_q is defined by

$$D_q f(z) = \frac{f(qz)-f(z)}{qz-z},\ 0<\left|q\right|<1,\ 0\neq z\in\mathbb{C}.$$

The derivative of the product of two functions and the derivative of the division of two functions are given by the following equation in [8] respectively

$$D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(qz)D_q\left(f(z)\right) - f(qz)D_q g(z)}{g(z)g(qz)}, \tag{1}$$

$$D_q\left(f(z)g(z)\right) = f(qz)D_q g(z) + g(z)D_q f(z).$$

The above q-standard notation can be found in [8]. Carlitz was the first to extend the classical Bernoulli polynomials, Euler numbers and polynomials, introducing them as q-Bernoulli and q-Euler numbers and polynomials ([1], [2], [3]). Srivastava et al. ([20], [21], [22], [23], [24], [25]) generalized the Bernoulli polynomials and Euler polynomials. In addition they investigated and proved some theorems for these polynomials and Apostol-Bernoulli and Apostol-Euler polynomials. Kim in ([9], [10]) gave some recursion relation for the q-Bernoulli and q-Euler polynomials. Furthermore, he proved some identities for the Frobenius-Euler polynomials. Srivastava-Pintér in [25] proved Srivastava-Pintér addition theorems. Kurt et al. in ([11], [12]) introduced the Frobenius-Euler polynomials and they proved some relations for these polynomials. Tremblay et al. in [28] generalized the new class of generalized Apostol-Bernoulli and Apostol-Euler polynomials. Also some mathematicians gave an analogue of the Srivastava-Pintér addition theorems.

Choi et al.[6] investigated q-Euler and q-Bernoulli polynomials and gave the relation between these polynomials, Choi et al.[7] proved some relation for the Apostol-Euler polynomials and Apostol-Bernoulli polynomials, Srivastava et al. [27] proved some relation for q-Bernoulli polynomials and multiple q-Zeta function.

Finally, Mahmudov in ([16], [17]) by using q-quantum calculus, introduced and gave some relations for the q-Bernoulli polynomials and q-Euler polynomials with two variable x, y.

In this work, we introduce q-Apostol type Frobenius-Euler polynomials. We give some new identities for the q-Apostol type Frobenius-Euler polynomials. Also, we prove some explicit expressions.

Definition 1.1. *Let $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$ and $0 < \left|q\right| < 1$. The q-Bernoulli numbers $\mathcal{B}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions:*

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t) - 1} \right)^{\alpha}, \ |t| < 2\pi \tag{2}$$

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t) - 1} \right)^{\alpha} e_q(tx) E_q(ty), \ |t| < 2\pi. \tag{3}$$

Definition 1.2. *Let $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$ and $0 < \left|q\right| < 1$. The q-Euler numbers $\mathcal{E}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions:*

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1} \right)^{\alpha}, \ |t| < \pi \tag{4}$$

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1} \right)^{\alpha} e_q(tx) E_q(ty), \ |t| < \pi. \tag{5}$$

Classical Frobenius-Euler polynomials $\mathcal{H}_n^{(\alpha)}(x;u)$ of order α are defined by the following relation ([1], [9], [11], [12])

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(x;u) \frac{t^n}{n!} = \left(\frac{1-u}{e^t - u} \right)^{\alpha} e^{xt} \tag{6}$$

where $\alpha \in \mathbb{N}$, u is an algebraic number.

Similarly, the Apostol type Frobenius-Euler polynomials $\mathcal{H}_n^{(\alpha)}(x;u;\lambda)$ of order α are defined by the following relation ([18])

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(x;u;\lambda) \frac{t^n}{n!} = \left(\frac{1-u}{\lambda e^t - u} \right)^{\alpha} e^{xt}. \tag{7}$$

Definition 1.3. *We define Apostol type q-Frobenius-Euler polynomials $\mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda)$ of order α in x, y and Apostol type q-Frobenius-Euler numbers $\mathcal{H}_{n,q}^{(\alpha)}(0,0;u;\lambda)$ of order α, respectively by*

$$\sum_{n=0}^{\infty} \mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda) \frac{t^n}{[n]_q!} = \left(\frac{1-u}{\lambda e_q(t) - u} \right)^{\alpha} e_q(tx) E_q(ty), \tag{8}$$

$$\sum_{n=0}^{\infty} \mathcal{H}_{n,q}^{(\alpha)}(0,0;u;\lambda) \frac{t^n}{[n]_q!} = \left(\frac{1-u}{\lambda e_q(t) - u} \right)^{\alpha}. \tag{9}$$

It is obvious that

$$\mathcal{H}_{n,q}^{(\alpha)} = \mathcal{H}_{n,q}^{(\alpha)}(0,0;u;\lambda), \ \lim_{q \to 1^-} \mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda) = \mathcal{H}_n^{(\alpha)}(x+y;u;\lambda)$$

$$\lim_{q \to 1^-} \mathcal{H}_{n,q}^{(\alpha)} = \mathcal{H}_{n,q}^{(\alpha)}(0,0;u;\lambda)$$

By this motivation, we define q-Apostol type Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y;\lambda)$ of order α and q-Apostol type Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y;\lambda)$ of order α respectively by

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)}(x,y;\lambda) \frac{t^n}{[n]_q!} = \left(\frac{t}{\lambda e_q(t) - 1} \right)^{\alpha} e_q(tx) E_q(ty) \tag{10}$$

and

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(\alpha)}(x,y;\lambda)\frac{t^n}{[n]_q!}=\left(\frac{2}{\lambda e_q(t)+1}\right)^{\alpha}e_q(tx)E_q(ty). \tag{11}$$

2. Some Basic Properties for the Apostol type q-Frobenius-Euler Polynomials

Proposition 2.1. *The following relations are true:*

$$\mathcal{E}_{n,q}(0,0;\lambda)=\frac{2}{\lambda+1}\mathcal{H}_{n,q}\left(0,0,(-\lambda)^{-1},1\right), \tag{12}$$

$$\mathcal{E}_{n,q}(x,y;\lambda)=\frac{2}{\lambda+1}\mathcal{H}_{n,q}\left(x,y(-\lambda)^{-1},1\right), \tag{13}$$

$$\mathcal{B}_{n,q}(x,y;\lambda)=\frac{1}{\lambda-1}[n]_q\mathcal{H}_{n-1,q}\left(x,y;\left(-\lambda^{-1}\right),1\right). \tag{14}$$

Proposition 2.2. *Apostol type Frobenius-Euler polynomials satisfy the following relations*

$$\mathcal{H}_{n,q}^{(\alpha+\beta)}(x,y;u;\lambda)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\mathcal{H}_{k,q}^{(\alpha)}(x,y;u;\lambda)\mathcal{H}_{n-k,q}^{(\beta)}(0,0;u;\lambda), \tag{15}$$

$$\lambda\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\mathcal{H}_{k,q}(x,y;u;\lambda)-\mathcal{H}_{n,q}(x,y;u;\lambda)=(1-u)(x+y)_q^n, \tag{16}$$

$$\mathcal{H}_{n,q}^{(\alpha-\beta)}(x,y;u;\lambda)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\mathcal{H}_{k,q}^{(\alpha)}(x,0;u;\lambda)\mathcal{H}_{n-k,q}^{(-\beta)}(0,y;u;\lambda). \tag{17}$$

Proposition 2.3.

$$D_{q,x}\left(\mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda)\right)=[n]_q\mathcal{H}_{n-1,q}^{(\alpha)}(x,y;u;\lambda), D_{q,t}e_q(tx)=xe_q(tx),$$

$$D_{q,y}\left(\mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda)\right)=[n]_q\mathcal{H}_{n-1,q}^{(\alpha)}(x,qy;u;\lambda), D_{q,t}E_q(ty)=yE_q(q+y).$$

Proof. The proof of these **Propositions** can be found from (2)-(11). □

Theorem 2.4. *There is the following recurrence relation for the Apostol type q-Frobenius-Euler polynomials*

$$\begin{aligned}&\mathcal{H}_{n+1,q}(x,y;u;\lambda)\\&=y\mathcal{H}_{n,q}(qx,qy;u;\lambda)+x\mathcal{H}_{n,q}(x,y;u;\lambda)-\lambda\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\mathcal{H}_{k,q}(x,y;u;\lambda)q^k\mathcal{H}_{n-k,q}(1,0;u,\lambda).\end{aligned} \tag{18}$$

Proof. For $\alpha=1$, in (7), we take the q-Jackson derivative of the Apostol type q-Frobenius-Euler polynomials $\mathcal{H}_{n,q}(x,y;u;\lambda)$ according to t.

$$\begin{aligned}\sum_{n=0}^{\infty}D_{q,t}\mathcal{H}_{n,q}(x,y;u;\lambda)\frac{t^n}{[n]_q!}&=D_{q,t}\left[(1-u)\frac{e_q(tx)E_q(ty)}{\lambda e_q(t)-u}\right]\\&=(1-u)D_{q,t}\left(\frac{e_q(tx)E_q(ty)}{\lambda e_q(t)-u}\right)\end{aligned}$$

By applying the equality (1) to the last expression, we have

$$= (1-u)\,\frac{\left(\lambda e_q(tq)-u\right)D_{q,t}\left[e_q(tx)E_q(ty)\right]-e_q\left(qtx\right)E_q\left(qty\right)D_{q,t}\left(\lambda e_q(t)-u\right)}{\left(\lambda e_q(t)-u\right)\left(\lambda e_q(qt)-u\right)}$$

$$= y\sum_{n=0}^{\infty}\mathcal{H}_{n,q}(qx,qy;u;\lambda)\frac{t^n}{[n]_q!}+x\sum_{n=0}^{\infty}\mathcal{H}_{n,q}(x,y;u;\lambda)\frac{t^n}{[n]_q!}$$

$$-\lambda\sum_{n=0}^{\infty}\mathcal{H}_{n,q}(x,y;u;\lambda)q^n\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathcal{H}_{n,q}\left(1,0;u;\lambda\right)\frac{t^n}{[n]_q!}.$$

By using Cauchy product, comparing the coefficient of $\frac{t^n}{[n]_q!}$, we have (18). □

Theorem 2.5. *There is the following relation for the generalized Apostol type q-Frobenius-Euler polynomials*

$$(2u-1)\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathcal{H}_{k,q}(0,0;u;\lambda)\mathcal{H}_{n-k,q}(x,y;1-u;\lambda)$$
$$= u\mathcal{H}_{n,q}(x,y;u;\lambda)-(1-u)\,\mathcal{H}_{n,q}(x,y;1-u;\lambda). \tag{19}$$

Proof. By using the identity

$$\frac{2u-1}{\left(\lambda e_q(t)-u\right)\left(\lambda e_q(t)-(1-u)\right)}=\frac{1}{\lambda e_q(t)-u}-\frac{1}{\lambda e_q(t)-(1-u)},$$

$$(2u-1)\,\frac{(1-u)\,e_q(xt)\,(1-(1-u))\,E_q(ty)}{\left(\lambda e_q(t)-u\right)\left(\lambda e_q(t)-(1-u)\right)}$$
$$=\frac{(1-u)\,e_q(xt)uE_q(ty)}{\lambda e_q(t)-u}-\frac{(1-u)\,e_q(xt)\,(1-(1-u))\,E_q(ty)}{\lambda e_q(t)-(1-u)},$$

$$(2u-1)\sum_{n=0}^{\infty}\mathcal{H}_{n,q}(0,0;u;\lambda)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathcal{H}_{n,q}(0,0;1-u;\lambda)\frac{t^n}{[n]_q!}$$
$$= u\sum_{n=0}^{\infty}\mathcal{H}_{n,q}(x,y;u;\lambda)\frac{t^n}{[n]_q!}-(1-u)\sum_{n=0}^{\infty}\mathcal{H}_{n,q}(x,y;1-u;\lambda)\frac{t^n}{[n]_q!}.$$

Comparing the coefficient of $\frac{t^n}{[n]_q!}$, we prove (19). □

Remark 2.6. *For* $\lim\limits_{q\longrightarrow 1^-}\mathcal{H}_{n,q}(x,y;u;\lambda)$. *substituting* $\lambda=1$, $y=0$ *in (19), we have Carlitz result ([1], equation 2.19).*

Theorem 2.7. *There is the following relation for the generalized Apostol type q-Frobenius-Euler polynomial*

$$u\mathcal{H}_{n,q}(x,y;u;\lambda)$$
$$=\lambda\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathcal{H}_{k,q}(x,y;u;\lambda)-(1-u)\,(x+y)_q^n\,. \tag{20}$$

Proof. By using the identity $e_q(t)E_q(-t) = 1$,

$$\frac{u}{\lambda\left(\lambda e_q(t) - u\right)e_q(t)} = \frac{1}{\left(\lambda e_q(t) - u\right)} - \frac{1}{\lambda e_q(t)}.$$

We write as

$$\frac{u\,(1-u)\,e_q(tx)E_q(yt)}{\left(\lambda e_q(t) - u\right)\lambda e_q(t)} = \frac{(1-u)\,e_q(tx)E_q(yt)}{\lambda e_q(t) - u} - \frac{(1-u)\,e_q(tx)E_q(yt)}{\lambda e_q(t)},$$

$$\frac{u}{\lambda}\sum_{n=0}^{\infty}\mathcal{H}_{n,q}(x,y;u;\lambda)\frac{t^n}{[n]_q!}\frac{1}{e_q(t)} = \sum_{n=0}^{\infty}\mathcal{H}_{n,q}(x,y;u;\lambda)\frac{t^n}{[n]_q!} - \frac{1-u}{\lambda e_q(t)}e_q(tx)E_q(yt),$$

$$\frac{u}{\lambda}\sum_{n=0}^{\infty}\mathcal{H}_{n,q}(x,y;u;\lambda)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\mathcal{H}_{n,q}(x,y;u;\lambda)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!} - \left(\frac{1-u}{\lambda}\right)\sum_{n=0}^{\infty}(x+y)_q^n\,\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, we have

$$u\mathcal{H}_{n,q}(x,y;u;\lambda) = \lambda\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \mathcal{H}_{k,q}(x,y;u;\lambda) - (1-u)\,(x+y)_q^n\,.$$

□

3. Explicit Relation for the Apostol type q-Frobenius-Euler Polynomials

Theorem 3.1. *There is the following relation for the Apostol type Frobenius-Euler polynomials*

$$\mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda) = \frac{1}{1-u}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \left\{\lambda\mathcal{H}_{k,q}(1,y;u;\lambda) - u\mathcal{H}_{k,q}(0,y;u;\lambda)\right\}\mathcal{H}_{n-k,q}^{(\alpha)}(x,0;u;\lambda). \tag{21}$$

Proof. Since (9)

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda)\frac{t^n}{[n]_q!} = \left(\frac{1-u}{\lambda e_q(t) - u}\right)^{\alpha} e_q(tx)E_q(ty) = \frac{1-u}{\lambda e_q(t) - u}E_q(ty)\frac{\lambda e_q(t) - u}{1-u}\left(\frac{1-u}{\lambda e_q(t) - u}\right)^{\alpha} e_q(tx)$$

$$= \frac{1}{1-u}\left\{\frac{1-u}{\lambda e_q(t)-u}E_q(ty)\lambda e_q(t)\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} e_q(tx)\right.$$
$$\left. -u\left(\frac{1-u}{\lambda e_q(t)-u}\right)E_q(ty)\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} e_q(tx)\right\}$$

$$= \frac{1}{1-u}\left\{\lambda\sum_{k=0}^{\infty}\mathcal{H}_{k,q}(1,y;u;\lambda)\frac{t^k}{[k]_q!}\sum_{l=0}^{\infty}\mathcal{H}_{l,q}^{(\alpha)}(x,0;u;\lambda)\frac{t^l}{[l]_q!} - u\sum_{k=0}^{\infty}\mathcal{H}_{k,q}(0,y;u;\lambda)\frac{t^k}{[k]_q!}\right.$$
$$\left.\times\sum_{l=0}^{\infty}\mathcal{H}_{l,q}^{(\alpha)}(x,0;u;\lambda)\frac{t^l}{[l]_q!}\right\}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, we have (21). □

Theorem 3.2. *There is the following relation between Apostol type q-Frobenius-Euler polynomials and the generalized Apostol q-Bernoulli polynomials*

$$\mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda)$$
$$= \frac{1}{[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix} n+1 \\ r\end{bmatrix}_q\sum_{k=0}^{r}\begin{bmatrix} r \\ k\end{bmatrix}_q\mathcal{B}_{n+1-r,q}(x,0;\lambda)\right.$$
$$\left.-\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k\end{bmatrix}_q\mathcal{B}_{n+1-k,q}(x,0;\lambda)\right\}\mathcal{H}_{k,q}^{(\alpha)}(0,y;u;\lambda). \tag{22}$$

Proof.

$$\left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} e_q(tx)E_q(ty)$$
$$= \left(\frac{1-u}{\lambda e_q(t)-u}\right)^{\alpha} E_q(ty)\frac{t}{\lambda e_q(t)-1}\frac{\lambda e_q(t)-1}{t}e_q(tx),$$

$$= \frac{1}{t}\left\{\lambda\sum_{n=0}^{\infty}\sum_{k=0}^{r}\begin{bmatrix} r \\ k\end{bmatrix}_q\sum_{r=0}^{n}\begin{bmatrix} n \\ r\end{bmatrix}_q\mathcal{H}_{k,q}^{(\alpha)}(0,y;u;\lambda)\mathcal{B}_{n-r,q}^{(\alpha)}(x,0;\lambda)\right.$$
$$\left.-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q\mathcal{H}_{k,q}^{(\alpha)}(0,y;u;\lambda)\mathcal{B}_{n-k,q}^{(\alpha)}(x,0;\lambda)\right\}\frac{t^n}{[n]_q!},$$

$$= \sum_{n=0}^{\infty}\frac{1}{[n+1]_q}\left\{\lambda\sum_{r=0}^{n+1}\begin{bmatrix} n+1 \\ r\end{bmatrix}_q\sum_{k=0}^{r}\begin{bmatrix} r \\ k\end{bmatrix}_q\mathcal{B}_{n+1-r,q}(x,0;\lambda)\right.$$
$$\left.-\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k\end{bmatrix}_q\mathcal{B}_{n+1-k,q}(x,0;\lambda)\right\}\mathcal{H}_{k,q}^{(\alpha)}(0,y;u;\lambda)\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, we have (22). □

Corollary 3.3. *There is the following relation between Apostol type q-Frobenius-Euler polynomials and the generalized Apostol q-Euler polynomials*

$$\mathcal{H}_{n,q}^{(\alpha)}(x,y;u;\lambda)$$
$$= \frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q\left\{\lambda\sum_{r=0}^{n}\begin{bmatrix} n \\ r\end{bmatrix}_q\mathcal{E}_{n-r,q}(x,0;\lambda)+\mathcal{E}_{n-k,q}(x,0;\lambda)\right\}\mathcal{H}_{k,q}^{(\alpha)}(0,y;u;\lambda)$$

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